NONLINEAR GALERKIN METHODS FOR EVOLUTION EQUATIONS WITH LIPSCHITZ CONTINUOUS STRONGLY MONOTONE OPERATORS

GÜNTER ALBINUS

WIAS — Preprint No. **124** *(1994)*

Weierstrass Institute for Applied Analysis and Stochastics

Mohrenstraße 39
D–10117 Berlin
Germany

Fax: + 49 30 2004975

1991 *Mathematics Subject Classification.* 47H05, 65M55, 47H20, 65M60.

Key words and phrases. nonlinear Galerkin methods, hierarchical bases, incremental unknowns

.

1. Introduction

Marion and Temam [2], [3] recommend nonlinear Galerkin methods for the initial value problem of evolution equations of the type

$$\dot{v} + \nu A v + R(v) = f \in H,$$

where A is a linear unbounded positive definite self-adjoint operator in a Hilbert space H such that there is a Gelfand triple $V \subset H \subset V'$ associated with the operator A. The nonlinear operator $R(u) = B(u) + Cu$ consists of a quadratic term $B(u)$ and a linear operator $C \in \mathcal{L}(V, H)$. Operators of this type with a positive number ν appear in the Navier-Stokes equations. The Galerkin approximation of the Cauchy problem in a large space $F \subset V$ get a block structure if the large space F is splitted up into a direct sum $F = G \oplus E$. In the finite-element case, e.g., F and G are the finite-element spaces on a fine or coarse grid, respectively, but the complement E is defined by the hierarchical basis. As in the case of multigrid methods it may be attractive to solve a cheaper variant of the equation on the fine grid, but the more expensive original problem on the coarse grid, i.e.

$$\dot{w} + \nu A w + R_G(u + w) = f_G, \quad w(0) = P_G v_0$$

on the space G, but a modified equation

$$\nu A_E u + \tilde{R}(w + u) = f_E$$

on the space E. In [2], [3] the structure of $\nu A + R$ is used rather intensively to show the 'equivalence' of the expensive original problem in the large space F with a cheaper modified coupled problem on $G \oplus E$.

In this paper we describe nonlinear Galerkin methods for evolution problems with nonlinear strongly monotone Lipschitz continuous operators. It will be shown that the solution of the Cauchy problem for the original operator can be approximated this way, but, in general, we can not prove the strong convergence in $L_2([0,T], V)$ as it holds for the usual Galerkin approximations in this case. Concerning monotone operators we refer to [1], but the reader is briefly remembered to some definitions or facts on monotone operators.

Let $(V, \|.\|)$ be a real separable reflexive Banach space continuously embedded into a Hilbert space H such that $V \subset H$ is a dense set. The scalar product and the norm in H are denoted by $(.,.)$ and $|.|$, respectively. The natural duality pairing on $V' \times V$ is denoted by $\langle .,. \rangle$. Let $S = [0,T] \quad (T > 0)$ be any compact interval. It is well known that the space $\mathbf{W}(S) = \{v \in L_2(S,V) : \quad \dot{v} \in L_2(S,V')\}$ with the norm $\|v\|_W = \|v\|_{L_2(S,V)} + \|\dot{v}\|_{L_2(S,V')}$ is a Banach space, which is continuously embedded into $C(S,H)$.

Let $A : V \mapsto V'$ be a Lipschitz continuous strongly monotone operator, i.e. there are constants $0 < m < M$ such that the estimates

$$\|A(v) - A(w)\|_{V'} \le M\|v - w\| \quad \text{and} \quad \langle A(v) - A(w), v - w \rangle \ge m\|v - w\|^2$$

$(v, w \in V)$ hold. The operator generates a Lipschitz continuous strongly monotone operator $\mathcal{A} : L_2(S,V) \mapsto L_2(S,V')$ by $(\mathcal{A}v)(t) = A[v(t)]$ a.e. on S. This operator has the same monotonicity constant and Lipschitz constant as A has. The following theorem is well known (cf. [1], chap VI, Theorem 1.1).

Theorem 1.1. *The Cauchy problem*

$$\dot{v} + \mathcal{A}v = f, \quad v(0) = a \tag{1.1}$$

with given elements $f \in L_\infty(R_+, V')$ and $a \in H$ has a unique solution u on R_+ which belongs to $\mathbf{W}(S) \subset C(S, H)$ for any S.

2. Finite-element spaces and their hierarchical splitting

Let $\Omega \subset R^2$ denote a bounded polygonal domain. We consider the space $V = H_0^1[\Omega \cup (\partial\Omega \setminus \Gamma_D)]$ of functions $v \in H^1(\Omega)$ which vanish on a fixed relatively closed subset $\Gamma_D \subset \partial\Omega$ of positive measure. The norm in V is the usual H_0^1 norm, $\|v\|^2 = \int_\Omega (\nabla v)^2$. The Hilbert space H is the space $L_2(\Omega)$ and thus $|.|$ denotes the usual L_2 norm. We consider triangulations τ of Ω which are Γ_D compatible, i.e. which satisfy the following conditions: There is a nonempty subset $\tau_D \subset \tau$ such that

1. each triangle $T \in \tau_D$ has one and only one side on Γ_D,
2. $\Gamma_D = \bigcup_{T \in \tau_D} \bar{T} \cap \Gamma_D$,
3. each triangle $T \in \tau \setminus \tau_D$ intersects Γ_D at most in a single point.

Let v_T denote the vector (v_i, v_j, v_k) of the values of a linear function v on a triangle $T = (ijk)$ in the vertices of T. Then we have

$$\|v\|_{L_2(T)}^2 = |T| v_T \cdot S_0 \cdot v_T' \quad \text{and} \quad \|v\|_{H_0^1(T)}^2 = |T| v_T \cdot S_T \cdot v_T'$$

with the area $|T|$ of the triangle and with the matrices

$$S_0 = \frac{1}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad 2|T| S_T = \begin{pmatrix} s_j + s_k & -s_k & -s_j \\ -s_k & s_k + s_i & -s_i \\ -s_j & -s_i & s_i + s_j \end{pmatrix},$$

where $s_l = \cot \theta_l$ with the angle θ_l of T at the vertex l. We consider families $\mathcal{T}$ of triangulations τ which have the following properties:

1. If $\tau \in \mathcal{T}$ then $\tau' \in \mathcal{T}$, where τ' consists of all triangles $T_l' \quad (l = i, j, k, c)$ which are generated by dividing each side of $T = (ijk) \in \tau$ in two equal parts;
2. there is a constant $C_\mathcal{T} \geq 1$ such that
$$c_\tau := \frac{\max_{T \in \tau} |T|}{\min_{T \in \tau} |T|} \leq C_\mathcal{T} \qquad (\tau \in \mathcal{T}); \textit{and}$$
3. there is a constant $\eta_\mathcal{T} \in]0, 1[$ such that
$$\max_{T=(ijk) \in \tau} \max_{l=i,j,k} \cos \theta_l \leq 1 - \eta_\mathcal{T} \qquad (\tau \in \mathcal{T}).$$

Families of triangulations which fulfill these conditions are called *regular*. For elementary domains there are triangulations τ_0 consisting of congruent triangles only. In such special cases a regular family $\mathcal{T}_0$ with $C_{\mathcal{T}_0} = 1$ is generated by uniform refinement of τ_0 according to (1). More general, any family $\mathcal{T}_0$ which is generated by uniform refinement of a finite triangulation τ_0 according to (1) is regular.

Sometimes we indicate the grid constant $h = \max_{T \in \tau} |T|^{1/2}$ by $\tau = \tau_h$. For a triangulation $\tau = \tau_h$ the standard finite-element space of continuous functions on $\bar{\Omega}$ which are linear on each $T \in \tau$ is denoted by $V_\tau = V_h$. We consider the decomposition $V_h = V_{2h} \oplus W_h$ with respect to the two-grid hierarchical basis. If τ_h' is the triangulation which is generated from a triangulation $\tau_{2h} \in \mathcal{T}$ according to (1) and if $\tilde{w}_T = (w_i, w_j, w_k)$ denotes the vector of the values of a function $w \in W_h$ in the midpoints on the sides of $T = (ijk) \in \tau_{2h}$ opposite to the vertices (remember that the functions $w \in W_h$ vanish in the grid points of τ_{2h}) then we have

$$|w|^2 = \sum_{T \in \tau_{2h}} |T| \tilde{w}_T \cdot S_1 \tilde{w}_T' \quad \text{and} \quad \|w\|^2 = \sum_{T \in \tau_{2h}} |T| \tilde{w}_T \cdot \tilde{S}_T \tilde{w}_T',$$

where

$$S_1 = \frac{1}{24} \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad 2|T|\tilde{S}_T = \begin{pmatrix} s & -s_k & -s_j \\ -s_k & s & -s_i \\ -s_j & -s_i & s \end{pmatrix}$$

with $s = s_i + s_j + s_k$. The property (3) of a regular family $\mathcal{T}$ implies the *strengthened Schwarz inequality*

$$|\int \nabla u \cdot \nabla w| \leq \sqrt{1-\eta}\|u\|\|w\| \quad (u \in V_{2h},\ w \in W_h) \tag{2.1}$$

(cf. [3]). The property (3) also implies that there is a constant $\lambda_{\mathcal{T}} > 0$ such that all eigenvalues λ_i^T of $S_0^{-1/2}|T|S_T S_0^{-1/2}$ $(T \in \tau \in \mathcal{T})$ are bounded by $\lambda_i^T \leq \lambda_{\mathcal{T}}$ and that there is a constant $\mu_{\mathcal{T}} > 0$ such that all eigenvalues μ_i^T of $S_1^{-1/2}|T|\tilde{S}_T S_1^{-1/2}$ $(T \in \tau \in \mathcal{T})$ are bounded by $\mu_i^T \geq \mu_{\mathcal{T}}$, since the positive semidefinite matrices $|T|S_T$ as well as the positive definite matrices $|T|\tilde{S}_T$ depend analytically on the parameters $s_i,\ s_j,\ s_k$ which vary in a compact set. These observations are summarized in the following

Lemma 2.1. *Let $\mathcal{T}$ be a regular family of triangulations with the constants $C_{\mathcal{T}},\ \lambda_{\mathcal{T}}$ and $\mu_{\mathcal{T}}$. Then for any triangulation $\tau_{2h} \in \mathcal{T}$ the estimates*

$$2h\|u\|_{H_0^1} \leq C_{\mathcal{T}}\sqrt{\lambda_{\mathcal{T}}}\|u\|_{L_2} \qquad (u \in V_{\tau_{2h}} \equiv V_{2h}), \tag{2.2}$$

$$\|w\|_{L_2} \leq \frac{2h}{\sqrt{\mu_{\mathcal{T}}}}\|w\|_{H_0^1} \qquad (w \in W_h) \tag{2.3}$$

($v = u + w \in V_{2h} \oplus W_h = V_h$) hold.

3. The formulation of the problem. Main result

We return to abstract spaces $V \subset H \cong H' \subset V'$ with the norms $\|.\|,\ |.|$ and $\|.\|_{V'}$, respectively, and a strongly monotone Lipschitz continuous operator $A : V \mapsto V'$. With regard to the preceding section, however, we additionally assume that

1. V is a Hilbert space with the scalar product $((.,.))$,
2. there are families $(V_h)_{h>0}$ and $(W_h)_{h>0}$ of finite dimensional linear subspaces $W_h \subset V_h \subset V$ such that $\bigcup_{h>0} V_h \subset V$ is dense, $V_h = V_{2h} \oplus W_h$ and that the following estimates hold:

$$\tilde{s}(h)\|u\| \leq C_0|u| \qquad (u \in V_{2h}), \tag{3.1}$$

$$|w| \leq \tilde{s}(h)\|w\| \qquad (w \in W_h) \tag{3.2}$$

with a constant $C_0 > 0$ and with a function $\tilde{s}(h) \to 0$ as $h \to 0$ and

$$((u,w)) \leq \sqrt{1-\eta}\|u\|\|w\| \qquad (u \in V_{2h},\ w \in W_h) \tag{3.3}$$

with some $\eta \in]0,1[$.

Let us denote

$$B^w(u) := A(u+w)|_{V_{2h}} \quad \text{and} \quad C^u(w) := A(u+w)|_{W_h}.$$

We note that the operators B^w and C^u are strongly monotone and Lipschitz continuous with the same constants m and M as A (maybe even better ones). Instead of the Galerkin approximation on V_h of the Cauchy problem (1.1) we consider the following system which consists of a Cauchy problem

$$\dot{u} + \mathcal{B}^w(u) = e, \quad u(0) = P_{2h}a \tag{3.4}$$

and the equations

$$C^{u(t)}[w(t)] = g(t) \tag{3.5}$$

for almost all $t \in S$. Here $e \in L_\infty(R_+, V'_{2h})$ and $g \in L_\infty(R_+, W'_h)$ denote the restrictions of f onto $L_2(S, V_{2h})$ or $L_2(S, W_h)$, respectively, and P_{2h} denotes the orthogonal projection of H onto V_{2h}. We ask for functions $u \in L_2(S, V_{2h})$ which satisfy $\dot{u} := \frac{d}{dt}u \in L_2(S, V'_{2h})$. Such functions belong to $C(S, V_{2h})$.

We observe that the system (3.4), (3.5) can be interpreted as the Cauchy problem

$$\dot{u} + \mathcal{E}_g(u) = e, \quad u(0) = P_{2h}a \tag{3.6}$$

with a radially continuous strongly monotone Volterra operator $\mathcal{E}_g : L_2(S, V_{2h}) \mapsto L_2(S, V'_{2h})$. Indeed, let $g \in W'_h$ be any fixed functional. The equation $C^u(w) = g$ has an uniqely determined solution $w = S_g(u)$ for any $u \in V_{2h}$, since the Operator C^u is strongly monotone and Lipschitz continuous.

Lemma 3.1. *The operator*

$$u \mapsto E_g(u) := B^{S_g(u)}(u) = A[u + S_g(u)]|_{V_{2h}}$$

with a given $g \in W'_h$ is radially continuous and strongly monotone. The strong monotonicity holds uniformly with respect to $g \in W'_h$.

Proof. We prove the strong monotonicity at first. Let $u_i \in V_{2h} \quad (i = 1,2)$ be given and let $w_i \in W_h$ denote the corresponding solutions . Then we have with $u = u_1 - u_2$ and with $w = w_1 - w_2$

$$\begin{aligned}
\langle A(u_i + w_i), w\rangle &= \langle g, w\rangle \qquad (i = 1,2),\\
\langle E_g(u_1) - E_g(u_2), u\rangle &= \langle A(u_1+w_1) - A(u_2+w_2), u\rangle\\
= \langle A(u_1+w_1) - A(u_2+w_2), u\rangle &+ \langle A(u_1+w_1) - A(u_2+w_2), w\rangle\\
= \langle A(u_1+w_1) - A(u_2+w_2) &, \ u_1 + w_1 - u_2 - w_2\rangle\\
\geq m\|u_1 + w_1 - (u_2+w_2)\|^2 &= m\|u_1 - u_2 + w_1 - w_2\|^2\\
\geq m(\|u_1-u_2\|^2 + \|w_1-w_2\|^2 &- 2\sqrt{1-\eta}\|u_1-u_2\|\|w_1-w_2\|)\\
&\geq m(1-\sqrt{1-\eta})\|u_1-u_2\|^2.
\end{aligned}$$

We remember that E_g is radially continuous if for any two elements $u, v \in V_{2h}$ the function $s \mapsto \langle E_g(u+sv), v\rangle$ is continuous on $[0,1]$. We consider fixed elements $u, v \in V_{2h}$ and a convergent sequence $s_j \to s_0$ in $[0,1]$. Let w_j denote the solutions of the equations $A(u+s_jv+w_j)=g$ on W_h for $j=0,1,2,\ldots$. The w_j satisfy the a priori estimate

$$\|w_j\| \le \frac{1}{m}(\|f-A(0)\|_{V'} + M\|u\| + M\|v\| \sup_k s_k),$$

since they are solutions of $A(u+s_jv+w)-A(u+s_jv)=g-A(u+s_jv)$ on W_h and since A is strongly monotone and Lipschitz continuous. Thus the set $\{w_j : j=1,2,\ldots\}$ of solutions is bounded in W_h, i.e. the sequence (w_j) contains a convergent subsequence $w_{j_k} \to w^*$, since W_h has finite dimension. Because of the Lipschitz continuity of A we have

$$g = A(u+s_{j_k}v+w_{j_k}) \to A(u+s_0v+w^*)$$

on W_h, i.e. $w^*=w_0,\ w_j \to w_0$ and

$$E_g(u+s_jv) = A(u+s_jv+w_j) \to A(u+s_0v+w_0) = E_g(u+s_0v),$$

i.e. in particular,

$$\langle E_g(u+s_jv), v\rangle \to \langle E_g(u+s_0v), v\rangle.$$

□

For $g \in L_\infty(R_+, W_h')$ and $u \in L_2(S, V_{2h})$ the integrand in

$$\langle \mathcal{E}_g(u), v\rangle = \int_0^T \langle E_{g(t)}[u(t)], v(t)\rangle dt = \langle \mathcal{A}(u+w), v\rangle$$

is defined for almost all $t \in S$. A priori estimates like above provide an integrable upper bound for $|\langle E_{g(t)}[u(t)+s_jv(t)], v(t)\rangle|$ such that Lebesgue's convergence theorem can be applied. Thus we have seen that $\mathcal{E}_g$ is a radially continuous strongly monotone operator. Now the Lemma 1.2 in [1], chap. VI can be applied, i.e.

Lemma 3.2. *The Cauchy problem* (3.6) *has exactly one solution u_h in $W(S) \cap L_2(S, V_{2h})$. The set $\{u_h : h > 0\}$ is bounded in $C(S,H)$ and in $L_2(S,V)$, and the set $\{\mathcal{A}(u_h+w_h) : h>0,\ w_h = S_g(u_h)\}$ is bounded in $L_2(S,V')$.*

Our main result is

Theorem 3.1. *The system* (3.4), (3.5) *is uniquely solvable for any $h > 0$. The first component u_h of its solution (u_h, w_h) is just the solution of the Cauchy problem* (3.6). *For any finite time interval $S=[0,T]$ it converges to the solution v of the Cauchy problem strongly in $C(S,H)$ and weakly in $L_2(S,V)$. The family of the w_h is bounded in $L_\infty(R_+,H)$, converges to 0 strongly in $L_2(S,H)$, weakly in $L_2(S,V)$ and weakly* in $L_\infty(R_+,H)$ as $h \to 0$. Furthermore, $\mathcal{A}(u_h+w_h)$ converges weakly to $\mathcal{A}(v)$ in $L_2(S,V')$.*

Remark 3.1. The operator A can be substituted by a family $(A(t))_{t\ge 0}$ of operators $A(t)=A(t,.)$ which are uniformly strongly monotone and uniformly Lipschitz continuous and which satisfies the condition $A(.,0) \in L_\infty(R_+,V')$. All assertions except the boundedness in $L_\infty(R_+,V')$ and the weak* convergence in this space remain valid, if we only assume that $f,\ A(.,0) \in L_{2,loc}(R_+,V')$ and that the strong monotonicity and the Lipschitz continuity hold uniformly on each bounded interval S.

Remark 3.2. The solutions (u_h, w_h) of the system (3.4), (3.5) for strongly monotone Lipschitz continuous operators A seem to be not so good as the usual Galerkin approximations (v_h) of (3.6) are. We were not able to prove

$$\int_0^T \langle A[u_h(t)+w_h(t)] - A[v(t)], u_h(t)+w_h(t)-v(t)\rangle dt \to 0 \quad \text{as} \quad h \to 0 \tag{3.7}$$

$(T > 0)$ in general, meanwhile the corresponding convergence for the v_h is well known. Nevertheless, it might be worthwhile to look for strongly monotone Lipschitz continuous operators having the property (3.7). In this case the following two corollaries would be applicable.

Corollary 3.1. *If* (3.7) *is satisfied then the sequences* $u_h \to v$, $w_h \to 0$ *and* $\mathcal{A}(u_h + w_h) \to \mathcal{A}(v)$ *strongly converge in* $L_2(S,V)$ *or in* $L_2(S,V')$, *respectively, as* $h \to 0$.

Proof. According to the assumption the $u_h + w_h$ strongly converge to v in $L_2(S,V)$. Therefore $u_j + w_j := u_{h_j} + w_{h_j}$ form a Cauchy sequence in $L_2(S,V)$ for any sequence $h_j \to 0$. The strengthened Schwarz inequality (3.3) implies the estimate

$$\begin{aligned} (3.8) \quad & \|u_j + w_j - u_k - w_k\|^2_{L_2(S,V)} = \\ & \|u_j - u_k\|^2_{L_2(S,V)} + \int_0^T 2((u_j(t)-u_k(t), w_j(t)-w_k(t)))dt + \|w_j - w_k\|^2_{L_2(S,V)} \\ \geq\ & \|u_j - u_k\|^2_{L_2(S,V)} - 2\sqrt{1-\eta}\int_0^T \|u_j-u_k\|_{L_2(S,V)}\|w_j-w_k\|_{L_2(S,V)}dt + \|w_j - w_k\|^2_{L_2(S,V)} \\ & \geq (1-\sqrt{1-\eta})(\|u_j-u_k\|^2_{L_2(S,V)} + \|w_j-w_k\|^2_{L_2(S,V)}), \end{aligned}$$

i.e. both the sequences (u_j) and (w_j) are strong Cauchy sequences in $L_2(S,V)$. Therefore the weak convergences $u_h \rightharpoonup v$ and $w_h \rightharpoonup 0$ in $L_2(S,V)$ imply even the strong convergences. □

Corollary 3.2. *Let* $\tilde{v}_h$ *denote the solution of the Galerkin approximation of the Cauchy problem* (1.1) *with respect to the subspace* V_h. *If* (3.7) *is satisfied then the error estimates*

$$\begin{aligned} |u_h(t) - \tilde{v}_h(t)|^2 &\leq \frac{M^2}{m}\int_0^t \exp\left[-\frac{m}{c_V^2}(t-s)\right]\|w_h(s)\|^2 ds, \\ |u_h(t) - v(t)|^2 &\leq |a - P_{2h}a|^2 \exp\left[-\frac{m}{c_V^2}t\right] \\ &+ \frac{M^2}{m}\int_0^t \exp\left[-\frac{m}{c_V^2}(t-s)\right]\|w_h(s)\|^2 ds \end{aligned}$$

hold.

Proof. Let $\tilde{v}$ denote either $\tilde{v}_h$ or the solution v of the Cauchy problem (1.1). Choosing $u_h - \tilde{v}$ as a test function one gets

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}[|u_h(t)-\tilde{v}(t)|^2] &+ \langle A[u_h(t)+w_h(t)] - A[\tilde{v}(t)], u_h(t)-\tilde{v}(t)\rangle = 0,\\
\frac{1}{2}\frac{d}{dt}[|u_h(t)-\tilde{v}(t)|^2] &+ \langle A[u_h(t)] - A[\tilde{v}(t)]\ ,\ u_h(t)-\tilde{v}(t)\rangle\\
&= \langle A[u_h(t)] - A[u_h(t)+w_h(t)]\ ,\ u_h(t)-\tilde{v}(t)\rangle,\\
\frac{1}{2}\frac{d}{dt}|u_h(t)-\tilde{v}(t)|^2 &+ m\|u_h(t)-\tilde{v}(t)\|^2 \le M\|w_h(t)\|\|u_h(t)-\tilde{v}(t)\|\\
&\le \frac{M}{2}[\frac{m}{M}\|u_h(t)-\tilde{v}(t)\|^2 + \frac{M}{m}\|w_h(t)\|^2].
\end{aligned}$$

Integrating the inequality

$$\frac{d}{dt}[|u_h(t)-\tilde{v}(t)|^2] + \frac{m}{c_V^2}|u_h(t)-\tilde{v}(t)|^2] \le \frac{M^2}{m}\|w_h(t)\|^2$$

one gets the assertion. □

The right-hand sides of the two error estimates converge to 0 with $h \to 0$ for each t, since w_h tends strongly to 0 in $L_2(S,V)$.

Remark 3.3. The basis of the nonlinear Galerkin method for strongly Lipschitz continuous operators A is the decomposition of the Galerkin spaces $V_h \subset V$ into almost orthogonal subspaces. The operator in the nonlinear Galerkin approximation is also strongly monotone, but with a worse constant depending on the factor $\sqrt{1-\eta}$ in the strengthened Schwarz inequality (3.3). As the Cauchy problem (3.6) and its Galerkin approximations are uniquely solvable for radially continuous coercive monotone operators, the nonlinear Galerkin approximation for the Cauchy problem (1.1) might be of interest for such operators, too.

4. Proof of the theorem 3.1

The proof follows closely the proofs of the theorems 1.1 and 1.2 in [1], chap. VI, and of the lemmata there. A crucial role in our proof plays switching from $\mathcal{E}_g$ to $\mathcal{A}(u+w)$ which is realized by adding the identity $\langle \mathcal{A}(u+w), w\rangle = \langle f, w\rangle$ or something like that in some places.

Let $u = u_h$ denote the solution of (3.6), $w = w_h$ the solution of (3.5) with the solution u as the parameter, and let $\tilde{m} = m(1-\sqrt{1-\eta})$ be the monotonicity constant of the $E_{g(t)}$. We observe that $E_{g(t)}(0) = A[w_0(t)]|_{V_{2h}}$, where w_0 denotes the solution of (3.5) with the parameter $u = 0$. Then the following estimates or identities hold:

$$\|w_0(t)\| \leq \frac{1}{m}(\|f\|_{L_\infty(R_+,V')} + \|A(0)\|_{V'}),$$

$$\|E_{g(t)}(0)\|_{V'_{2h}} \leq \|A[w_0(t)]\|_{V'} \leq \frac{M}{m}(\|f\|_{L_\infty(R_+,V')} + \|A(0)\|_{V'}), \tag{4.1}$$

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \tilde{m}\|u(t)\|^2 \leq [(1+\frac{M}{m})\|f\|_{L_\infty(R_+,V')} + \frac{M}{m}\|A(0)\|_{V'}]\|u(t)\|, \tag{4.2}$$

(4.3)

$$\frac{1}{2}|u(T)|^2 + \int_0^T \langle\, E_{g(t)}[u(t)] - E_{g(t)}[0], u(t)\,\rangle = \frac{1}{2}|a|^2 + \int_0^T \langle f(t) - E_{g(t)}(0), u(t)\rangle,$$

$$\|u\|^2_{L_2(S,V)} \leq \frac{|a|^2}{\tilde{m}} + \frac{1}{\tilde{m}^2}\|f - A(w_0)\|^2_{L_2(S,V')} =: C_1, \tag{4.4}$$

$$\|\mathcal{E}_g(u)\|_{L_2(S,V'_{2h})} \leq \|\mathcal{A}(u+w)\|_{L_2(S,V')} \leq C_2 \qquad (w = S_g(u)), \tag{4.5}$$

$$\|w(t)\| \leq \frac{1}{m}[\|f(t) - A(0)\|_{V'} + M\|u(t)\|] \quad \text{a.e. on } R_+, \tag{4.6}$$

$$\|w\|_{L_2(S,V)} \leq \frac{1}{m}\|f - A(0)\|_{L_2(S,V')} + \frac{M}{m}\|u\|_{L_2(S,V)} \leq C_3. \tag{4.7}$$

Let c_V denote the constant of the continuous embedding $V \subset H$. Integrating the consequence

$$\frac{d}{dt}|u(t)|^2 + \frac{\tilde{m}}{c_V^2}|u(t)|^2 \leq \frac{2}{\tilde{m}}C_4$$

of (4.2) and (4.1) one gets

$$|u(t)|^2 \leq |a|^2 \exp(-\frac{\tilde{m}}{c_V^2}t) + \frac{2c_V^2}{\tilde{m}^2}[1 - \exp(-\frac{\tilde{m}}{c_V^2}t)]C_4, \tag{4.8}$$

with

$$C_4 = (1 + 2\frac{M^2}{m^2})\|f\|^2_{L_\infty(R_+,V')} + 2\frac{M^2}{m^2}\|A(0)\|^2_{V'},$$

i.e. $u \in L_\infty(R_+, H)$. Combining (3.1) and (3.2) with (4.6) and (4.8) one gets $w \in L_\infty(R_+, H)$. So far all estimates of this section hold uniformly with respect to the omitted parameter h. Combining (3.1) and (3.2) with (4.7) one gets the strong convergence $w_h \to 0$ in $L_2(S, H)$.

The strong convergence of the w_h in $L_2(S, H)$ in connection with the boundedness of $\{w_h : \ h > 0\}$ in $L_2(S, V)$ and in $L_\infty(R_+, H)$ also imply the weak convergence $w_h \rightharpoonup 0$ in $L_2(S, V)$ and the weak* convergence $w_h \rightharpoonup^* 0$ in $L_\infty(R_+, H)$ as $h \to 0$.

Convergence of u_h. The boundedness of $\{u_h : h > 0\}$ in $L_2(S, V)$ and in $L_\infty(R_+, H)$, the uniform (with respect to t) boundedness of $\{u_h(t) : h > 0\}$ in H and the boundedness of $\{\mathcal{A}(u_h + w_h) : h > 0\}$ in $L_2(S, V')$ imply the existence of a sequence

$h_j \to 0$ such that

$$\begin{aligned} u_j \equiv u_{h_j} \rightharpoonup u^* &\quad \text{in} \quad L_2(S,V), \\ u_j \to u^* \text{ weak}^* &\quad \text{in} \quad L_\infty(R_+,H), \\ u_j(t) \rightharpoonup \tilde{u}(t) &\quad \text{in} \quad H \quad \text{for all } t \in S \text{ and} \\ \mathcal{A}(u_j + w_j) \rightharpoonup \xi &\quad \text{in} \quad L_2(S,V'). \end{aligned}$$

Choosing test functions $b\phi$, $b \in \bigcup_{h>0} V_{2h}$ and $\phi \in C_0^\infty(\mathrm{int}S)$ and $(T-t)b$ one proves

$$u^* \in W(S),\ u^*(0) = a,\ u^*(t) = \tilde{u}(t), \text{ and } \dot{u}^* + \xi = f.$$

as in [1], chap. VI, Lemma 1.4. We observe

$$\langle \mathcal{A}(u_j + w_j), u_j + w_j \rangle = \langle \mathcal{E}_g(u_j), u_j \rangle + \langle f, w_j \rangle = \langle f, u_j + w_j \rangle - \langle \dot{u}_j, u_j \rangle;$$

Then we get the estimate

$$\begin{aligned} \overline{\lim} \langle \mathcal{A}(u_j + w_j), u_j + w_j \rangle &= \overline{\lim} [\langle f, u_j + w_j \rangle + \frac{1}{2}(|u_j(0)|^2 - |u_j(T)|^2)] \\ &\leq \langle f, u^* \rangle + \frac{1}{2}(|a|^2 - |u^*(T)|^2)] = \langle f - \dot{u}^*, u^* \rangle = \langle \xi, u^* \rangle, \end{aligned}$$

which provides $\dot{u}^* + \mathcal{A}(u^*) = f$ according to [1], chap.III, Lemma 1.3.

Since the Cauchy problem (1.1) has an unique solution v, we have $v = u^*$, u_h and $u_h + w_h \rightharpoonup v$ in $L_2(S,V)$ as well as $\mathcal{A}(u_h + w_h) \rightharpoonup \mathcal{A}(v)$ in $L_(S,V')$. Let $(v_h)_{h>0}$ be a family of functions $v_h \in C^1(S, V_{2h})$ converging to v in $\mathbf{W}(S)$. Then

$$\begin{aligned} \frac{1}{2}|u_h(t) - v_h(t)|^2 - \frac{1}{2}|u_h(0) - v_h(0)|^2 &= \int_0^t \langle \dot{u}_h - \dot{v} + \dot{v} - \dot{v}_h, u_h - v_h \rangle \\ = \int_0^t \langle -A(u_h + w_h) + A(v)\dot{v} - \dot{v}_h, u_h - v_h \rangle &= \int_0^t \langle -A(u_h + w_h) + A(v), u_h + w_h - v \rangle + \\ \int_0^t \langle -A(u_h + w_h) + A(v), u_h + w_h - v \rangle &+ \int_0^t \langle \dot{v} - \dot{v}_h, u_h - v_h \rangle \\ \leq \int_0^t \langle -A(u_h + w_h) + A(v), u_h + w_h - v \rangle &+ K_0 \|v - v_h\|_{L_2(S,V)} + K_1 \|\dot{v} - \dot{v}_h\|_{L_2(S,V')} \\ &\leq \int_0^t \langle -A(u_h + w_h) + A(v), u_h + w_h - v \rangle + \max_i K_i \|v - v_h\|_{\mathbf{W}(S)}. \end{aligned}$$

Since $\mathbf{W}(S) \subset C(S,H)$ is continuously embedded the estimates or limit values

$$\begin{aligned} \lim_{h\to 0} |u_h(0) - v_h(0)| &\leq \lim_{h\to 0}(|P_{2h}a - a| + |v(0) - v_h(0)|) = 0, \\ \lim_{h\to 0} \|u_h - v\|_{C(S,H)} &\leq \lim_{h\to 0}(\|v_h - v\|_{C(S,H)} + \|u_h - v_h\|_{C(S,H)}) = 0, \\ \lim_{h\to 0} \int_0^t \langle -A(u_h + w_h) + A(v) &, \quad u_h + w_h - v \rangle = 0 \end{aligned}$$

hold. This completes the proof of 3.1.

Corrigendum (during electronization). The convergence (3.7) holds such that the assertions of the Corollary 3.1 are true. The incorrect estimates of

$$\frac{1}{2}|u_h(t) - v_h(t)|^2 - \frac{1}{2}|u_h(0) - v_h(0)|^2$$

has to be substituted by the identity

$$
\begin{aligned}
\frac{1}{2}|u_h(t) - v_h(t)|^2 \quad &- \quad \frac{1}{2}|u_h(0) - v_h(0)|^2 \\
&= \quad \int_0^t \langle \dot{u}_h - \dot{v} + \dot{v} - \dot{v}_h, u_h - v_h \rangle ds \\
= \int_0^t \langle \dot{v} - \dot{v}_h, u_h - v_h \rangle ds \quad &- \quad \int_0^t \langle \mathcal{A}(u_h + w_h) - \mathcal{A}(v), u_h - v_h \rangle ds \\
= \int_0^t \langle \dot{v} - \dot{v}_h, u_h - v_h \rangle ds \quad &- \quad \int_0^t \langle \mathcal{A}(u_h + w_h) - \mathcal{A}(v), u_h + w_h - v_h \rangle ds \\
&+ \quad \int_0^t \langle f - \mathcal{A}(v), w_h \rangle ds \\
= \int_0^t \langle \dot{v} - \dot{v}_h, u_h - v_h \rangle ds \quad &- \quad \int_0^t \langle \mathcal{A}(u_h + w_h) - \mathcal{A}(v), u_h + w_h - v \rangle ds \\
+ \int_0^t \langle f - \mathcal{A}(v), w_h \rangle ds \quad &+ \quad \int_0^t \langle \mathcal{A}(u_h + w_h) - \mathcal{A}(v), v_h - v \rangle ds
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
0 \leq \frac{1}{2}|u_h(t) - v_h(t)|^2 \quad &+ \quad \int_0^t \langle \mathcal{A}(u_h + w_h) - \mathcal{A}(v), u_h + w_h - v \rangle ds \\
= \frac{1}{2}|u_h(0) - v_h(0)|^2 \quad &+ \quad \int_0^t \langle \dot{v} - \dot{v}_h, u_h - v_h \rangle ds \\
+ \int_0^t \langle f - \mathcal{A}(v), w_h \rangle ds \quad &+ \quad \int_0^t \langle \mathcal{A}(u_h + w_h) - \mathcal{A}(v), v_h - v \rangle ds.
\end{aligned}
$$

All the terms on the right-hand side converge uniformly with respect to $0 < t \leq T$ to 0 as $h \to 0$.

The statement of Corollary 3.2 holds for $\tilde{v}_h$ only, but not for v. The proof does not work for v, since $u_h - v$ can not be taken as a test function for the Galerkin approximation.

It should be still remarked that the class of bounded demicontinuous strongly monotone operators is more natural for the questions investigated in the preprint than the class of Lipschitz continuous strongly monotone operators.

References

[1] H. Gajewski, K. Gröger und K. Zacharias, Nichtlineare Operatorgleichungen. Berlin 1974.

[2] M. Marion and R. Temam, Nonlinear Galerkin methods. SIAM J. Numer. Anal. **26**(1989), 5, 1139-1157.

[3] M. Marion and R. Temam, Nonlinear Galerkin methods: The finite-elements case. Numer. Math. **57**(1990), 205-226.

Acknowledgement. I would like to thank Professor Herbert Gajewski, Professor Konrad Gröger and my colleague Klaus Zacharias for stimulating discussions and several helpful suggestions concerning the paper.

Günter Albinus, WIAS, Mohrenstrasse 39, D–10117 Berlin, Germany

E-mail address: albinus@wias-berlin.d400.de